A new theory for moist convection in statistical equilibrium

A. Parodi(1), K. Emanuel(2)

(2) CIMA Research Foundation, Savona, Italy
(3) EAPS, MIT, Boston, USA
“True” dynamics: turbulent, moist, non-Boussinesq, precipitating convection

Can we find a simplified dynamical model?
Which is the key process in moist deep convection?
Cloud and water vapour processes affect radiative transfer and the earth’s climate in several ways:

• by coupling dynamical and microphysical processes in the atmosphere through the heat of condensation and evaporation and through redistributions of sensible and latent heat and momentum;

• by coupling radiative and dynamical–microphysical processes in the atmosphere through the reflection, absorption, and emission of radiation;

• by influencing hydrological process in the ground through precipitation;

• by influencing the couplings between the atmosphere and oceans (or ground) through modifications of radiation and planetary boundary layer (PBL) processes.
It has been well documented that the extent to which GCMs (Global Circulation Models) can be used as reliable tools to study climate dynamics and change crucially depends on how adequately tropical convection and related water vapour processes can be represented in these models.

In this framework, one of the main factors limiting our understanding of cloud dynamics and water vapour control by deep convection is the lack of a correct scaling for velocity and buoyancy in moist convection, even though there have been several studies of this topic (Renno et al, 1994; Emanuel and Bister, 1996; Grabowski, 2003; Robe and Emanuel, 1996, 2001; Xu and Randall, 1998; Wu, 2002).
One of the most important issues in the study of dry and moist convection in statistical equilibrium is the determination of buoyancy and velocity scales. These are relevant to such issues as:

• the phase transition of random convective states to organized tropical cyclone-dominated convection

• study of the interaction between moist convective plumes and boundary layer convection

• the absence of a scaling theory for moist convection limits our understanding of the feedbacks of changes in large-scale forcing on convective ensembles and then the development of comprehensive representations of convection in large scale models.
In the absence of large-scale circulations, the tropical atmosphere would assume a state of **radiative-moist convective equilibrium**, in which the divergence of the net vertical radiative flux (shortwave and longwave) would be compensated by the convergence of the vertical flux of enthalpy in convective clouds, except for in a thin boundary layer next to the surface, in which ordinary dry turbulence would carry the flux.
The canonical problem of radiative-dry convective equilibrium was first developed by Prandtl (1910, 1942). This problem is the simplest model that captures some of the essential aspects of atmospheric convection.

\[ Q_{\text{rad}} \]

\[ T = \text{constant} \]
A simplified approach to atmospheric convection: The Prandtl problem (1925)

\[ F_{\text{top}} = 0 \]

Ideal stratosphere
- \( h \approx 5 \text{ Km} \)
- \( Q_{\text{rad}} = 0 \)

Ideal troposphere
- \( H \approx 10 \text{ Km} \)
- \( Q_{\text{rad}} < 0 \)

\[ F_{\text{bottom}} = c \left( T_S - T \bigg|_{z=0} \right) \]

The Prandtl problem

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + g \frac{\rho}{\rho_0} + v \nabla^2 u \]

\[ \frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T - \frac{Q_{rad}}{c_p} \]

\[ \nabla \cdot u = 0 \]

\[ \tilde{x} = x/H \]
\[ \tilde{t} = U t/H \]
\[ \tilde{u} = u/U \]
\[ \tilde{p} = p/\rho_0 U^2 \]
\[ \tilde{T} = (T - T_s)/\Delta T \]

\[ U = \left( \frac{g Q_0 H^2}{c_p T_s} \right)^{\frac{1}{3}} \]
\[ \Delta T = \frac{U^2 T_s}{g H} . \]

\[ F_{bottom} = -\kappa \frac{\partial T}{\partial z} = c(T_s - T |_{z=0}) \quad \text{for } z = 0 \]

\[ \frac{\partial T}{\partial z} = 0 \quad \text{for } z = D \]

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + T \hat{k} + \frac{1}{Re} \nabla^2 u \]

\[ \frac{\partial T}{\partial t} + u \cdot \nabla T = \frac{1}{Re Pr} \nabla^2 T - \theta(z_0 - z) \]

\[ \nabla \cdot u = 0 \]

\[ u = v = w = 0 \quad \text{for } z = 0 \text{ and } z = 1 + \frac{h}{H} \]

\[ \frac{\partial T}{\partial z} = \frac{c}{U} Re Pr T \quad \text{for } z = 0 \]

\[ \frac{\partial T}{\partial z} = 0 \quad \text{for } z = 1 + \frac{h}{H} \]
The Prandtl problem

\( H_t^+ = \langle w\theta \rangle_{++} \) \ with \ \( w > 0, \ \theta > 0 \)
Vertical velocity: troposphere \((z=H/2)\)
Vertical velocity: troposphere \((z=H)\)
Vertical velocity: stratosphere ($z=0.75$)

SUBSIDENCE

PLUMES
A simple proxy for convective precipitation?

\[ P(x, y, z, t) \]

\[ H_t^+ = \langle w \theta \rangle_{+++} \quad \text{with} \quad w > 0, \ \theta > 0 \]

\[ P(x, y, t) = \int_{t-\Delta t}^{t} \int_{z_1}^{z_2} dz \int_{z_1}^{z_2} dt \ H_t^+ (x, y, z, t) \]
Comparison between Prandtl Problem and GATE statistics
Buoyancy and velocity scales for dry convection in statistical equilibrium were derived long ago by Prandtl (1910, 1925):
for this problem the turbulence kinetic energy scales as $(Fz)^{2/3}$, where $z$ is the altitude above the surface, while the unstable stratification decreases as $z^{-4/3}$

But the question of convective velocity and buoyancy scales, as well as the topic of fractional area coverage of convective clouds, are unresolved in moist convection (in radiative convective equilibrium) (Emanuel and Bister, 1996; Grabowski, 2003; Robe and Emanuel, 1996, 2001; Xu and Randall, 1998; Wu, 2001)
Computational domain

- Doubly periodic domain
- Constant cooling rate $Q_{rad}$ over the full height of the troposphere

$T = \text{constant}$
The radiative convective equilibrium problem: buoyancy and velocity scales

• High resolution (meso-γ scale) simulations of an atmosphere in radiative convective equilibrium are performed using WRF model;

• Prescribing different constant cooling rates, $Q_{\text{rad}}$, we try to characterize the velocity and buoyancy scales for moist convection in statistical equilibrium;

• The dependence of spatio-temporal properties of the convective flow field on numerical and physical details, such as domain size and microphysics, is investigated
• The underlying hypothesis of the work is that convective updraft velocities and rainfall intensity scale with the terminal velocity of raindrops;
• For prescribed cooling rate $Q_{\text{rad}}$, a set of simulations with different raindrop terminal velocities $V_T$ (range: 2-50 m/s) is performed and the results are compared.
Why do we use this microphysical parameter?

Dynamical forcing and/or orographic lifting acting on a saturated grid box may initiate the process.
Why do we use this microphysical parameter?

At first, $V_T$ can be considered as a proxy for the raindrop size, since bigger raindrops fall with higher terminal velocities, than is the case with small particles.

Second, variations in the raindrop terminal velocity can affect the heat balance of the subcloud layer and in the middle of the troposphere through the processes of evaporation, condensation and production of cloud water by accretion.

Third, Cifelli et al. (2000), among others, measured drop-size distribution characteristics in tropical mesoscale convective systems and showed strong variability of drop size distributions (DSD) parameters for stratiform or deep convective rainfall: stratiform events are dominated by small size, slowly falling raindrops, while deep convective processes produce larger raindrops.
What microphysical processes can grow precipitation particles quickly?

Coalescence: $T > 0 \text{ deg C}$

Aggregation: $T < 0 \text{ deg C}$

Riming

All these processes are function of $V_T$
To stay simple we consider:

warm rain scheme

constant cooling rate $Q_{\text{rad}}$: A constant cooling rate $Q_{\text{rad}}$ is applied from the surface to 15 km, while above 15 km a sponge layer relaxes the scalar state variables back to observed troposphere and stratosphere profiles (Jordan, 1958). Such an approach allows separating the impact of cloud microphysics on convective dynamics from the combined impact on convective dynamics and radiative transfer. Moreover, cloud microphysics is still capable of exerting significant impact on the mean temperature and moisture profiles even in the case where radiative impacts are neglected.
Back to moist scaling…

WRF Model settings

- Domain sizes: 25x25x20 km, 50x50x20 km, 100x100x20 km, 200x200x20 km and 400x400x20 km
- Horizontal resolution: 2 km (in the future 1 km is planned)
- No orography
- Doubly periodic boundary conditions
- Kessler microphysics
- Constant cooling rate $Q_{\text{rad}}$: -2 K/day, -4 K/day and -6 K/day ($0<z<15 \text{ km}$)
- Rayleigh damping layer in upper layers ($15<z<18 \text{ km}$)
- The lower boundary is a passive ocean with constant temperature $T_S=300 \text{ K}$
Radiative convective equilibrium statistics: role of the domain size

\[ Q_{\text{rad}} = -4 \text{ K/day}, \text{ raindrop terminal velocity } V_T = 5 \text{ m/s} \]

We take 200 km as the reference domain size.
Radiative convective equilibrium statistics: Horizontally averaged thermodynamic profiles

$Q_{\text{rad}} = -4 \text{ K/day, } V_T = 5 \text{ m/s: control run}$
Radiative convective equilibrium statistics: Horizontally averaged thermodynamic profiles

\[ Q_{\text{rad}} = -4 \text{ K/day: } \langle T_{VT} \rangle - \langle T_{VT=5 \text{ m/s}} \rangle \]

Mean temperature: as \( V_T \) increases the atmosphere becomes colder outside the BL (Grabowski, 2003)
Radiative convective equilibrium statistics:
Horizontally averaged thermodynamic profiles

\[ Q_{\text{rad}} = -4 \text{ K/day} \]

Mean water vapour: as \( V_T \) increases the atmosphere becomes drier (Grabowski, 2003)
Radiative convective equilibrium statistics: Vertical velocity CDF

$Q_{\text{rad}} = -4 \text{ K/day}$

$z = 5000 \text{ m}$

The total probability of $w > 2 \text{ m/s}$ is of the order of 1% while the bulk of the distribution lies at very low value ($<1 \text{ m/s}$)
Radiative convective equilibrium statistics: Buoyancy and velocity scales

- Convective rain cell identification: first we detect local maxima of the rainfall intensity field $R$. We retain only those maxima in $R$ that exceed the threshold of 30 mm/h. The horizontal extent of each rain cell is then determined by identifying the connected region around the maximum that has $R$ larger than 0.5 mm/h;

3. We evaluate the dependence of some statistical cell properties (mean and std dev of rainfall intensity and number of cells) on the raindrop terminal velocity.
Radiative convective equilibrium statistics: Buoyancy and velocity scales

Rainfall intensity scaling (over a period of 4 days in radiative convective equilibrium: 96 frames, time resolution =30 min)

Scaling of mean and standard deviation of rainfall intensity versus raindrop terminal velocity VT in the convective cell core (threshold for rain cell detection equal to 10 mm/h).
Radiative convective equilibrium statistics: Buoyancy and velocity scales

Rainfall intensity scaling (over a period of 4 days in radiative convective equilibrium: 96 frames, time resolution = 30 min)

Scaling of mean and standard deviation of rainfall intensity versus raindrop terminal velocity VT in the convective cell core (threshold for rain cell detection equal to 30 mm/h).
Scaling of mean value of maximum and minimum downdraft (over the whole domain) versus the raindrop terminal velocity VT.
Radiative convective equilibrium statistics: Buoyancy and velocity scales

Mean size of detected cells depends on the adopted threshold

As $V_T$ increases the intensity of rain cells grows and their size decreases and tends to converge.
The buoyancy of an air parcel lifted to some level can be broken into two parts: a part owing to temperature excess and a part owing to liquid water loading:

\[ B = g \frac{\alpha}{\alpha'} - gl\alpha' \]  

(1)

where \( \alpha \) is specific volume, \( l \) is the liquid water concentration, \( g \) is the acceleration of gravity, and the primes denote deviations (at constant pressure) from the base state, represented by an overbar. Expressing \( \alpha \) as a function of pressure and saturation moist entropy, \( s^* \), and using one of Maxwell’s relations, the above expression can be written:

\[ B = \Gamma s^* - gl \]  

(2a)
We suppose that fluctuations of saturation entropy in the cloud are directly related to fluctuations of actual entropy in the boundary layer, allowing us to re-write (2 a) as:

\[ B = \Gamma s - gl \]  

\[ (2b) \]

Therefore, the partitioning between thermal and condensed water loading results in the following expression for the kinetic energy due to vertical velocity component:

\[ \frac{1}{2} w^2 = E\Delta Ts' - gH_q l' \]  

\[ (3) \]

where \( \Delta T \) is the temperature difference across the depth of the convecting layer and \( H_q \) is a scale height for liquid water concentration.
Assuming that in the case of radiative-convective equilibrium the turbulence near the top of the planetary boundary layer is of convective origin, it can be stated that the turbulent flux of entropy at the surface scales with the imposed radiative integrated over the depth $H$ of the troposphere:

$$\overline{wB} = \dot{Q}H \quad (4)$$

The boundary layer similarity theory provides an approximate expression for the vertical velocity perturbation as a function of the surface sensible heat flux $Q_s$ and the boundary layer depth $h$:

$$w \equiv \left( hQ_s \right)^{1/3} \quad (5)$$

Moreover in statistical equilibrium the surface sensible heat flux must balance the imposed radiative cooling integrated across the boundary layer depth $h$. Then it is possible to formulate a scaling for the entropy fluctuations near the top of the planetary boundary layer:
Buoyancy and velocity scales: Scaling theory (4)

\[
sF = H \frac{\dot{Q}}{h}^{\frac{2}{3}} \tag{6}
\]

In order to estimate the subcloud layer depth, the bulk formulas for surface fluxes are used and the expressions for the sensible heat balance of the subcloud layer and the overall heat balance of the troposphere are written down:

\[
C_k | V | c_p \left( T_s - T_a \right) = \dot{Q} h \tag{7a}
\]

\[
C_k | V | L_v \left( q_s^* - q_a \right) = \dot{Q} H \tag{7b}
\]

where \( c_p \) is the heat capacity of air, \( V \) is the surface wind, \( L_v \) is the latent heat of vaporization, \( T_s \) is the sea surface temperature, \( T_a \) is the air temperature, \( q_a \) is the water vapor mass fraction of air and finally \( q_s^* \) the saturation specific humidity corresponding to saturation at sea surface temperature.
Buoyancy and velocity scales: Scaling theory (5)

The difference in specific humidity at sea level can be expressed by linearizing the Clausius-Clapeyron equation:

\[ q_s^* - q_a \cong \frac{L_v q_s^*}{R_v T_c^2} (T_s - T_a) + q_a (1 - RH) \quad (8) \]

A quasi-empirical relationship between the relative humidity deficit, RH, of surface air and the depth of the subcloud layer can be introduced:

\[ h \cong 2000 \, m (1 - RH) \quad (9) \]

Finally, substituting (7a) and (9) into (8), and the result into (7b) an expression for the subcloud layer depth \( h \) can be obtained. From that expression, it is possible to derive an expression for the scaling of the entropy fluctuations near the top of the planetary boundary layer:

\[ s' : \quad H^3 \left( \frac{0.8 L_v \dot{Q}}{H \rho R_v T_s^2} + \frac{C_k |V|^2}{2000 \, m} \right)^{\frac{2}{3}} \quad (10) \]
Assuming that not much of the rain water falls back through the updraft, the following expression provides a scale for the liquid water concentration, where $q^*$ is a characteristic difference between specific humidity in the cloud and outside the cloud:

\[
V_T l' \cong wq^* + wl \quad (11)
\]

\[
l' = \frac{wq^*}{V_T - w}
\]

Using (11) for $w$ and (10) for $T_s$ in (3) gives a cubic equation for $w$:

\[
\frac{1}{2} w^2 = \Delta Ts' - gH_q l' \quad (3)
\]

\[
\frac{1}{2} w^2 = \int \Delta Ts' - gH_q \frac{wq^*}{V_T - w}
\]

\[
\frac{1}{2} w^2 = \int C \Delta Ts' - gH_q \frac{wq^*}{A V_T - w}
\]
Buoyancy and velocity scales: Scaling theory (7)

\[
\begin{align*}
\frac{1}{2}w^3 - \frac{1}{2}w^2V_T - wH_q q^* + \Delta TH^3 V_q s^* + \frac{3}{2} 0.8L_v Q
\end{align*}
\]

\[
\frac{C_k |V|}{2000 m} V_T = 0 \quad (12)
\]
Figure 1: comparison between theory and model results for the maximum updraft case.
Buoyancy and velocity scales: Scaling theory (9)

Figure 2: comparison between theory and model results for the mean updraft case.
Conclusions

• Buoyancy and velocity scales for moist convection in statistical equilibrium are studied;

• A minimum domain size (around 200x200 km$^2$) to achieve a significant convergence in the statistical properties of the moist convection is identified;

• In agreement with Grabowski (2003) and others, we find that as $V_T$ increases the atmosphere becomes more convective and drier

• We find that convective updraft velocity and rainfall intensity scale with the raindrop terminal velocity

• A scaling theory is proposed and its validation looks like satisfactory

• Similar results have been found with another LAM, COSMO Model (DWD, Germany) and this contributes to strengthen the reliability of these findings

• Generalization of theory for quasi-equilibrium problem: introduction of a time-varying SST with different amplitudes and period
Acknowledgements

- Franco Siccardi (CIMA Research Foundation)
- Antonello Provenzale (ISAC-CNR)